Evolution of a sandpile in a thick-flow regime

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We solve a one-dimensional sandpile problem analytically in a thick flow regime when the pile evolution may be described by a set of linear equations. We demonstrate that, if an income flow is constant, a space periodicity takes place while the sandpile evolves even for a pile of only one type of particles. Hence, grains are piling layer by layer. The thickness of the layers is proportional to the input flow of particles r_0 and coincides with the thickness of stratified layers in a two-component sandpile problem, which were observed recently. We find that the surface angle θ of the pile reaches its final critical value (θ_f) only at long times after a complicated relaxation process. The deviation ($\theta_f - \theta$) behaves asymptotically as $(t/r_0)^{-1/2}$. It appears that the pile evolution depends on initial conditions. We consider two cases: (i) grains are absent at the initial moment, and (ii) there is already a pile with a critical slope initially. Although at long times the behavior appears to be similar in both cases, some differences are observed for the different initial conditions are observed. We show that the periodicity disappears if the input flow increases with time.

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I. INTRODUCTION

Granular flows have attracted increasing interest over the last years, and constitute now a very active research field with apparent technological applications [1–9]. Reasons for such great attractiveness of granular media or of the so called soft matter are clear: they present not only unusual properties that let them look like solids in some cases and liquids in other ones, but they also display new phenomena unknown both to solids and liquids. Such a new and intriguing phenomenon is recently observed spontaneous stratification of granular mixtures [10–16]. When a granular mixture is poured into a Hele-Shaw cell (i.e., in a quasi-two-dimensional silo) under some conditions, layers of different grains appear to be arranged periodically.

Several studies were carried out to explain the striking stratification [10–16] although the problem seems to be very complicated for an analytical treatment. In our communication [17] we put forward the following question: is it possible to find some precursor of stratification in a much simpler situation when only one type of grains is poured? We shall present a full answer below for one of the limit regimes of sandpile spilling using an approach most convenient for analytical consideration.

Crucial progress in understanding of granular flow nature was made some years ago when Bouchaud, Cates, Prakash, and Edwards [18] (see also the papers of Mehta and Barker [19] and of Mehta, Needs, and Dattagupta [20]) introduced a clear phenomenological theory that describes a surface flow of granular materials. The main idea of the approach is the following. The grains are divided to two parts—static grains and rolling ones. The conversion of static grains to rolling ones occurs depending on the relation between the local slope of a pile and the repose angle of the material. In addition, the continuity equation for the total amount of grains holds. Thus, the problem can be reduced to two coupled partial differential equations for the local amounts of static and rolling grains. One should note that similar ideas were also applied to a self-organized criticality problem [21–23].

Nevertheless, even this simple approach is still complicated because of nonlinearity of the equations. Recently, Boutreux, Raphaël and P.-G. de Gennes [24] proposed a phenomenological description of some special case of granular flows — a so called thick flow regime — that provides a unique possibility for an analytical treatment: in this case the coupled equations are linear and one of them is completely decoupled from the other one. In papers [24,25] this approach was used to study some granular flow configurations. Relaxation of sand from several most simple states was considered, generalization to many-component flows were proposed [26–28], and new improvements of the approach were made [29] to make it closer to reality, though the main classical sandpile problem still remained unsolved even for the simplest version of such a description.

We shall use the proposed phenomenological equations to answer positively the above stated question and to describe a total evolution of the sandpile, that will turn to be surprisingly complex. We will show that the slope of the pile approaches its critical value only at long times after a complicated discontinuous relaxation process [17]. In fact, we shall reconsider a classical sandpile problem using, maybe, the simplest possible idea without appealing to more refined approaches like, for instance, self-organized criticality [30,31,9].

To start with, let us write out the phenomenological equations we shall use. Boutreux, Raphaël, and P.-G. de Gennes [24] describe phenomenologically the one-dimensional thick granular flows (the flow thickness supposed to be much

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higher than the grain size) by the following equations:

$$\frac{\partial r}{\partial t} - v\frac{\partial r}{\partial x} = v_u\left(\frac{\partial h}{\partial x} - \theta_f\right),$$

$$\frac{\partial h}{\partial t} = -v_u\left(\frac{\partial h}{\partial x} - \theta_f\right). \tag{1}$$

Here $h(x,t)$ is a profile of the static part of the material; $r(x,t)$ is the width of a moving granular layer; we assume that the flow is from right to left and all rolling grains are supposed to move with an equal velocity v. $r(x,t)$ and $h(x,t)$ describe completely the evolution of a pile. The sum of Eqs. (1), $\partial(r+h)/\partial t - v\,\partial r/\partial x = 0$, has the form of the continuity equation for grains, $vr(x,t)$ is a local flow. The right hand parts describe the conversion of the static grains to rolling ones and vice versa depending on a relation between a local slope $\partial h/\partial x$ and θ_f. The meaning of v_u is the velocity of the uphill fronts as we shall see later. Usually $v_u > v$ [14,24]. θ_f is a critical angle, or a so called repose angle, that is the angle to which the sandpile will evolve. The deviations of the local slope from θ_f are supposed to be small. Eq. (1) is valid for $r(x,t) > 0$. If $r(x,t) = 0$, then the right hand parts of Eqs. (1) are supposed to be equal to zero. Therefore, the function $v_u(r)$, which equals const·r in the thin flow regime [18], now is a constant v_u for $r > 0$ and is zero at $r = 0$.

The physical reasons to introduce equations (1) in such a form are described in [24,25]. One can see that, at a given point, static grains converse to rolling ones only if the local slope of the pile is higher than the critical angle, and rolling grains converse to static ones, if the local slope is lower than the critical angle. These linear equations are much simpler for an analytical treatment than the previously proposed nonlinear equations for a thin flow regime [18,26–28] in which the characteristic velocity v_u in Eq. (1) is replaced by $r(x,t)$ divided by a constant with the dimensionality of time.

In fact, Eqs. (1) represent, maybe, a minimal model for the description of granular flows. The equations are really very simple: the second equation is independent of the first one. The general solution of Eqs. (1) may be written immediately in the form [24]:

$$r(x,t) = u(x+vt) - \frac{v_u}{v+v_u}w(x-v_ut) - v_u\theta_f t,$$

$$h(x,t) = w(x-v_ut) + v_u\theta_f t, \quad r > 0; \tag{2}$$

$$h(x,t) = \text{const}, \quad r = 0,$$

where $u(x)$ and $w(x)$ are arbitrary functions. We shall use it to describe the sandpile evolution phenomenologically neglecting, as usual [24,25], possible near-front deviations from the thick flow regime. In fact, in paper [25], moving fronts of granular flows were studied in intermediate situations between the thick flow regime and the thin flow regime. The results [25] show that such a neglect is possible.

It is convenient to use also the relation describing conservation of number of grains that follows from the continuity equation:

$$\int_{-\infty}^{0} dx[h(x,t) + r(x,t)]$$

$$= vr_0t + \int_{-\infty}^{0} dx[h(x,0) + r(x,0)]. \tag{3}$$

It follows directly from Eq. (2) that the solutions consist of the part $w(x-v_ut)$ moving with the velocity v_u to the right, the part $u(x+vt)$ moving with the velocity v to the left, and the growing (for h) or decreasing (for r) homogeneous background. Specific initial and boundary conditions, which we shall use in Secs. II and III, will result in $u(z)$ and $w(z)$ consisting only of linear parts. Therefore, the solutions will have breaks between linear parts. Some of them will move to the right with the velocity v_u and others to the left with the velocity v. Coordinates of left fronts of the rolling and static grain distributions coincide and, because of the decreasing linearly (for r) or increasing linearly (for h) homogeneous background, may move with velocities lower then v if the front of $r(x)$ is not of jumplike form. One may check, using, e.g., Eq. (3), that a jumplike front of the rolling grain distribution $r(x,t)$ may move only with velocity v to the left. As it follows from Eq. (2), linear parts of $r(x)$ and $h(x)$ can move only parallel to themselves or are motionless. Thus, local slopes of the distributions can change only discontinuously if an income flow is constant in time. We shall show the picture of the evolution more clearly for particular examples in Secs. II and III.

Thus, to describe the pile evolution we only have to solve linear equations with initial and boundary conditions. The boundary conditions are defined from the condition on an input flow (we set it constant usually) and from consideration of a moving front of the pile — a moving boundary. In fact, that is the most difficult point in solving problems of this kind.

There are several possibilities to choose initial conditions. In Secs. II and III we consider two most interesting and natural cases: (i) grains are absent at the initial moment, and (ii) there is already a pile with a critical slope initially. We shall show that the pile evolution is a complicated process in both cases. The slope of a pile approaches its critical value only at infinite times after a long relaxation process during which areas with different slopes are present. At long times, the pile evolution looks very similar for both initial conditions, although, some distinction can be found even at the infinity. Thus, we observe a long memory of initial conditions.

The results obtained for initial conditions (i) and (ii) are presented in Figs. 1 and 5 — movies of r and h distribution profiles, in Figs. 2 and 6 — diagrams of trajectories of fronts and breaks of the grain distributions on the t,x-plane, and in Figs. 3 and 7 — the dependencies of the pile slopes on time.

In fact, all these figures depict a long relaxation process to the critical state (i.e., to the pile with the critical angle) by different means. For example, from Fig. 3 one sees that the relaxation is discontinuous — local derivatives of the slope over time are zero or infinity for all times.

We shall demonstrate the space periodicity appearing while a pile evolves: it turns out that grains are piling layer by layer (see Fig. 2 and 6) and the thickness of the layers just
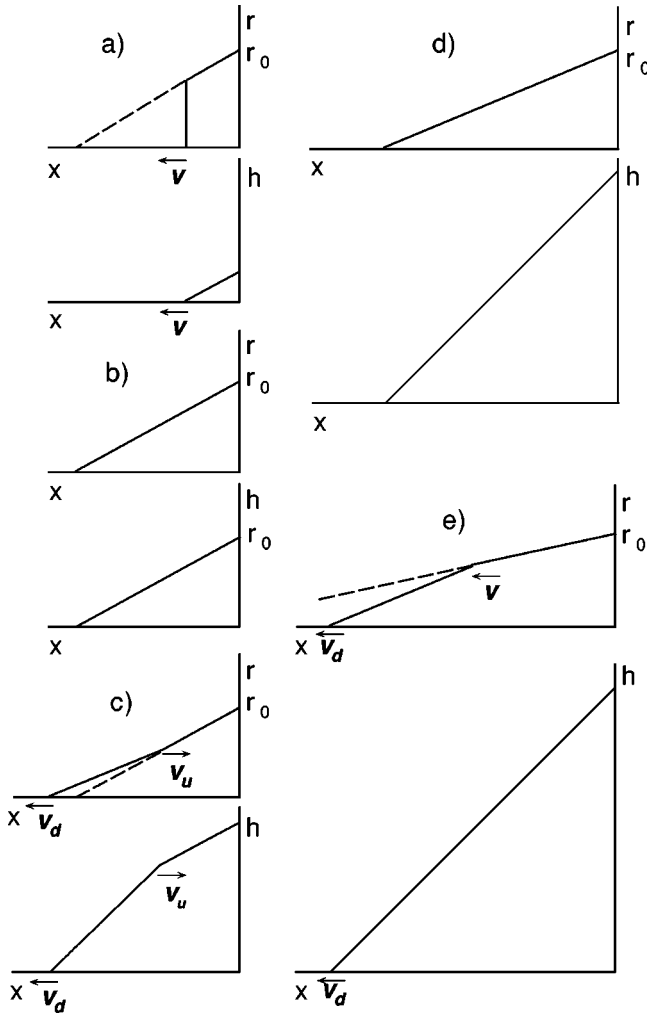
FIG. 1. The evolution of the profiles of rolling grains $r(x)$ and static grains $h(x)$ in the region $-\infty < x < 0$. In the initial state grains are absent, $r(-\infty < x < 0, t = 0) = h(-\infty < x < 0, t = 0) = 0$. (a) $0 < t < t_1$, the front moves with the velocity v. (b) $t = t_1$ [see Fig. 2 and Eq. (4)]. (c) $t_1 < t < t_2$, the front moves with some velocity $v_d < v < v_u$, the breaks of the profiles move uphill with the velocity v_u. (d) $t = t_2$, the breaks approach the wall at $x = 0$. (e) $t_2 < t < t_3$, the front proceeds to move with the velocity v_d, the break of $r(x)$ moves downhill with the velocity v. Note, that the right linear part of $r(x)$ is always motionless. After the break of $r(x)$ overtakes the front at $t = t_3$, the general configurations (b)—(e) is repeated with a lower front velocity.

coincides with the thickness of the stratified layers observed in two-component sandpiles [11,14]. Thus, we give positive answer to the question stated above and find a precursor of an intriguing stratification phenomena already in an one-component sandpile problem.

One may wonder now if there is any possibility for a sandpile to evolve without a space periodicity at all. We demonstrate that, of course, such evolution is possible in a thick flow regime, for example, if an input flow is a linear function of time. We show that the front moves with a constant velocity all the time, and linear profiles of moving and static grains expand without any surprises in that situation (see Fig. 4). The relation between amounts of static and rolling grains in a pile depends on the rate of the input flow increase.
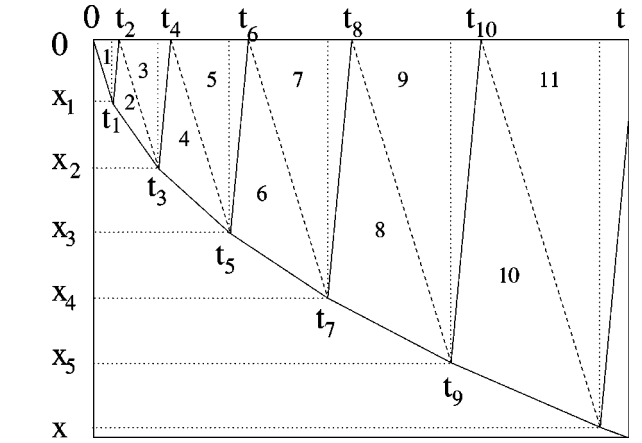

FIG. 2. The areas of different solutions for the profiles of rolling and static grains for the pile evolution starting from an empty state [see Eqs. (22) and (23)]. $v/v_u = 0.3$. The lower segmented line shows the dependence of the front coordinate on time. The solid lines $x = v_u(t - t_{2m})$, $m = 1, 2, \ldots$ depict the uphill movement of the $r(x)$ and $h(x)$ breaks. The dashed lines $x = -v(t - t_{2m})$ show the downhill movement of the break of the $r(x)$ profile. The points x_m are arranged periodically.

II. SANDPILE EVOLUTION STARTING FROM EMPTY STATE

A. Constant input flow

The case we will consider in this section corresponds to the situation when there are no particles in the initial state.

Let there be a wall at $x = 0$, and grains be poured permanently beginning from the moment $t = 0$ at this point, so $r(x = 0, t) = r_0$ is a boundary condition (r_0 is the thickness of the input flow). Let, first, r_0 be constant in time. The sandpile is supposed to expand to the left, i.e., to $x = -\infty$. There are no particles at the initial stage so $r(x \leq 0, t = 0) = 0$, $h(x \leq 0, t = 0) = 0$ are the initial conditions. Inserting general solutions from Eqs. (2) and the above initial and boundary conditions one may obtain the functions $u(z)$ and $w(z)$.

To clear up the picture and to simplify the calculations we start from the following consideration of the begininning stage. [Below, we shall demonstrate consistent complete calculations in the more frequent case, which is realized at more long times, of a $r(x)$ front linearly decreasing to zero.]

One may see that because of the initial and boundary conditions for $r(x, t)$ and of the form of general solution for it Eq. (2), $r(x, t)$ should first have a jump front; the part of $r(x, t)$ adjacent to $x = 0$ has to be independent of t. The last statement follows immediately from the form of the general solutions Eq. (2) and from the fixed right boundary condition $r(x = 0, t) = r_0$. The first statement follows from the last one — the jump front of $r(x, t)$ is the only possibility for the first stage of the process. But then, as we have seen in Sec. I, it has to move with velocity v.

The distribution of static grains $h(x, t)$, which appears from the precipitation of rolling ones has a front that goes to zero without an abrupt leap but linearly in the case under consideration, since there are no fixed boundary conditions for h in the right. Then one can easily imagine that at first the solutions have to look as shown in Fig. 1(a) (as we have noticed, nothing but linear functions may appear from the considered initial and boundary conditions). In principle, one
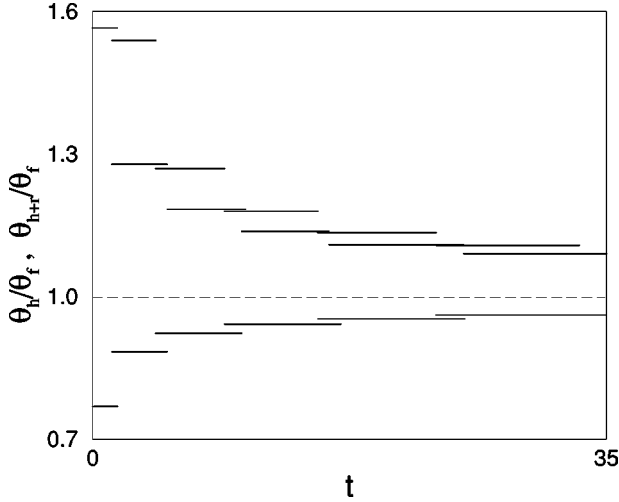
FIG. 3. The dependence of the relative pile slope on time when the pile evolution starts from an empty state. $v/v_u = 0.3$. θ_h is the slope of a static part of the pile. θ_{h+r} is the slope of the whole pile consisting of static and rolling parts. The upper set of lines shows θ_{h+r}. At infinity $\theta_h, \theta_{h+r} \rightarrow \theta_f$. The separated lines for θ_{h+r} are defined for $0 < t < t_2$, $t_1 < t < t_3$, $t_2 < t < t_4$, etc. (see Fig. 2). The lines for θ_h are defined for $0 < t < t_2$, $t_1 < t < t_4$, $t_3 < t < t_6$, etc. Sometimes (e.g., at $t_1 < t < t_2$, $t_3 < t < t_4$, etc. for θ_h) the profiles have two parts with different slopes.

may immediately insert r and h in this form $[r(x,t) = r_0 - ax$, $h(x,t) = b(x+vt)$, where a and b are constant coefficients] into Eq. (1) and obtain the following answer in the time interval $0 < t < t_1 = (v+v_u)r_0/(vv_u\theta_f)$:

$$r(x,t) = \left(r_0 + \frac{v_u}{v+v_u} \theta_f x \right) \Theta(x+vt)$$

$$\tag{4}$$

$$h(x,t) = \frac{v_u}{v+v_u} \theta_f (x+vt).$$

$\Theta(x)$ is the Heaviside function [we do not write the multiplier $\Theta(x+vt)$ in the right hand side part of the second equation of Eqs. (4), since, of course, $r(x)$ and $h(x)$ can not be negative]. The static grains are precipitated from the rolling ones, so the front coordinates of $r(x)$ and $h(x)$ should coincide. The meaning of the time t_1 is clear from Fig. 1(a) — it is the very first instant at which the jump at the front appears to be zero. Note, that the slope of the static part $\partial h/\partial x = v_u \theta_f/(v+v_u)$ is less than the critical slope, so the relaxation to the critical slope is nontrivial.

Nevertheless, to be sure, we prefer to substitute the general solutions Eq. (2) into the following set of initial and boundary conditions

$$r(x,t=0) = 0,$$

$$h(x,t=0) = 0,$$

$$\tag{5}$$

$$r(x=0, 0 < t < t_1) = r_0,$$

$$h(x=-vt, 0 < t < t_1) = 0.$$

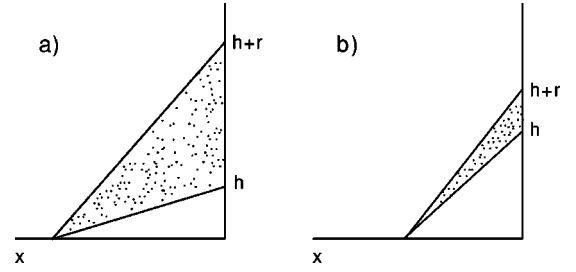Using previous considerations to simplify the calculations, we assume in Eq. (5) the front velocity to be equal to

FIG. 4. The evolution of a sandpile with a linearly growing input flow: $r_0 = v_0 t$, $v_0 = $ const. An initial state is empty. We show here both static and rolling grains in the same plots, so profiles for $h(x)$ and $h(x) + r(x)$ are presented. The fronts moves with a time-independent velocity v_d which is a function of v_0. (a) The rate v_0 is much higher than v_0^* [see Eq. (33) and the text]. The front velocity is close to its maximal possible value v, rolling grains have no time to converse into static ones, so the amount of rolling grains is much greater than the amount of static grains. (b) The rate v_0 is much lower than v_0^*. A relative amount of rolling grains is small, the front velocity tends to zero, and the slope of the distribution of static grains is nearly critical.

v from the very beginning, so a fifth condition for $r(x,t)$ at the front point is not necessary. In principle, the time t_1 may be obtained if one demands that all parts of functions $u(z)$ and $w(z)$ have to be connected together continuously, but we prefer to use the already known expression for t_1 and be sure that calculations are self-consistent [i.e., all parts of functions $u(z)$ and $w(z)$ are connected together continuously] only afterwards.

Substituting the general solutions Eq. (2) into Eq. (5) we get

$$w\left(-\frac{(v+v_u)^2}{vv_u} \frac{r_0}{\theta_f} < z < 0 \right) = \frac{v_u}{v+v_u} \theta_f z,$$

$$\tag{6}$$

$$u\left(0 < z < \frac{v+v_u}{v_u} \frac{r_0}{\theta_f} \right) = r_0 + \frac{v_u(v+2v_u)}{(v+v_u)^2} \theta_f z.$$

After substitution Eq. (6) into Eq. (2) we again obtain our solution Eq. (4) of Eq. (1) with a front moving to the left with the velocity v in the time interval $0 < t < t_1$ [see Fig. 1(a)].

As a result, at the time t_1 one gets

$$r(x,t_1) = r_0 + \frac{v_u}{v+v_u} \theta_f x,$$

$$\tag{7a}$$

$$h(x,t_1) = r_0 + \frac{v_u}{v+v_u} \theta_f x,$$

$$\tag{7b}$$

for $-[(v+v_u)/(vv_u)]r_0/\theta_f < x < 0$ [Fig. 1(b)]. These equations are used as initial conditions to find the solutions in the next time interval $t_1 < t < t_3$, at which the front of $r(x,t)$ will be jumpless.

Times t_2 and t_3 appear naturally from the solution (see Fig. 1) but we prefer to write out their expressions immediately: $t_2 = [(v+v_u)^2/vv_u^2]r_0/\theta_f$ and $t_3 = [(v+v_u)(v+3v_u)/vv_u^2]r_0/\theta_f$. As one may understand from the figure, the meaning of the times is the following: t_2 is the time at
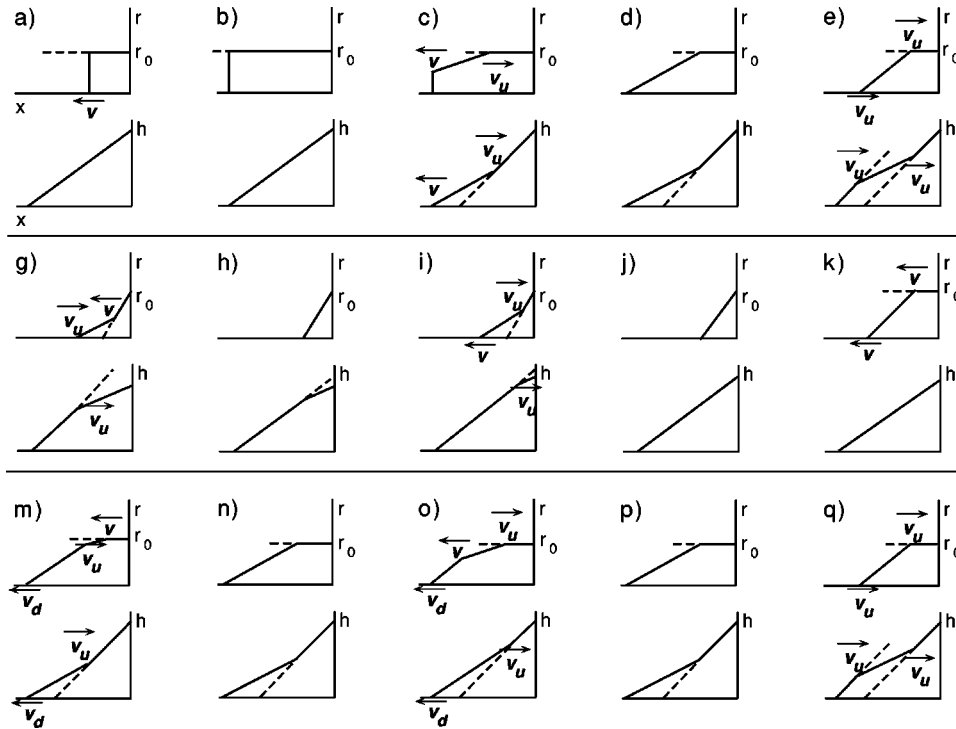
FIG. 5. The evolution of the profiles of rolling grains $r(x)$ and static grains $h(x)$ starting from the critical state (see the text for details). At the initial moment $r(-\infty<x<0,t=0)=0$ and $h(-\infty<x<0,t=0)=\theta_f(x+d)$. Plots (a)–(q) show the profiles at some successive instants. Afterwards the evolution proceeds in a similar way. The scale is changed from one plot to another.

which breaks of $r(x,t)$ and $h(x,t)$ moving from the front to the right with the velocity v_u will approach the point $x=0$; and t_3 is the time at which the break of $r(x,t)$ moving from $x=0$ will overtake the front.

To simplify our calculations, we again use the expression for t_3 and check the correctness of the choice of t_3 at the very end of the calculations.

Now we show how to treat moving boundary conditions. We have to add to the initial conditions Eq. (7) the following boundary conditions:

$$r(x=0,t>0)=r_0, \qquad (8a)$$

$$r\left(x=-\frac{v+v_u}{vv_u}\frac{r_0}{\theta_f}-v_d(t-t_1),t_1<t<t_3\right)=0, \qquad (8b)$$

$$h\left(x=-\frac{v+v_u}{vv_u}\frac{r_0}{\theta_f}-v_d(t-t_1),t_1<t<t_3\right)=0. \qquad (8c)$$

Equations (8b) and (8c) are the conditions for the left front of the pile that is supposed to move with a yet unknown velocity v_d. All we have to do is (i) to insert the general solutions Eq. (2) into Eqs. (7) and (8); (ii) to find v_d, $u(z)$, and $w(z)$; and (iii) to check the self-consistency of our choice of t_3, which was made beforehand, in fact, from
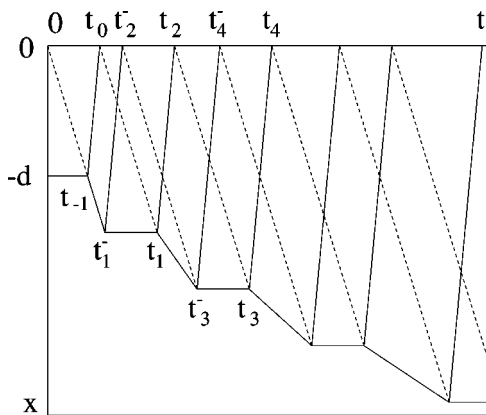

FIG. 6. Trajectories of the front and the breaks of the $r(x)$ and $h(x)$ profiles on t,x-plane in the case of the pile evolution starting from a critical state (compare with Fig. 2). $v/v_u=0.3$. $h(x,t=0)=\theta_f(x+d)$. The lower segmented line shows the dependence of the front coordinate on time. The solid lines $x=v_u(t-t_{2m}),m=1,2,\ldots$ depict the uphill movement of the $r(x)$ and $h(x)$ breaks. The dashed lines $x=-v(t-t_{2m})$ show the downhill movement of the break of the $r(x)$ profile. The coordinates of plateaus are arranged periodically.
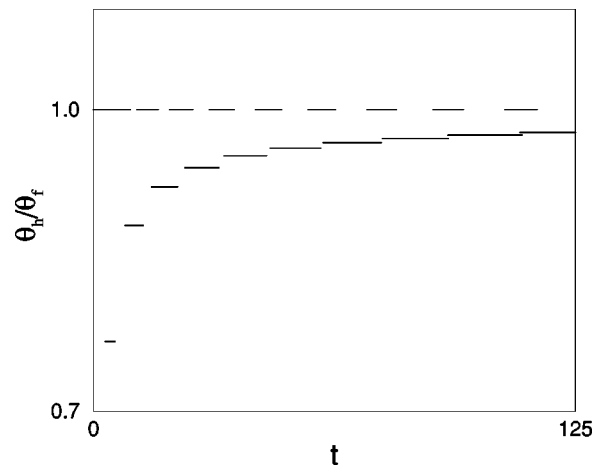

FIG. 7. The dependence of the relative pile slope θ_h/θ_f on time when the pile evolution starts from the critical state (compare with Fig. 3). $v/v_u=0.3$. In some time intervals the profile of static grains has two or even three parts with different slopes. Segments with the slope equal exactly to the critical one are present even at long times.

physical reasons. Note that the beforehand choice of t_3 was used only to make the following intermediate formulas more compact.

Substitution of Eq. (2) into Eq. (8c), (7b), (7a), and (8b) gives

$$w\left(-\frac{v+v_u}{vv_u^2}(v_dv+2v_uv+2v_dv_u+3v_u^2)\frac{r_0}{\theta_f}\right.$$

$$\left.<z<-\frac{(v+v_u)^2}{vv_u}\frac{r_0}{\theta_f}\right)$$

$$=\frac{(v+v_u)(v-v_d)}{v(v_u+v_d)}r_0+\frac{v_u}{v_u+v_d}\theta_fz, \qquad (9)$$

$$w\left(-\frac{(v+v_u)^2}{vv_u}\frac{r_0}{\theta_f}<z<-\frac{v+v_u}{v}\frac{r_0}{\theta_f}\right)=\frac{v_u}{v+v_u}\theta_fz, \qquad (10)$$

$$u\left(0<z<\frac{v+v_u}{v_u}\frac{r_0}{\theta_f}\right)=r_0+\frac{v_u(v+2v_u)}{(v+v_u)^2}\theta_fz, \qquad (11)$$

and

$$u\left(0<z<\frac{(v-v_d)(v+v_u)(v+2v_u)}{vv_u^2}\frac{r_0}{\theta_f}\right)$$

$$=r_0+\frac{vv_u}{(v-v_d)(v+v_u)}\theta_fz \qquad (12)$$

correspondingly. The functions $u(z)$ from Eqs. (11) and (12) have to coincide since they are defined at (at least) overlapping intervals of z (we shall see below that, in fact, these intervals coincide). Thus, equating Eqs. (11) and (12) we obtain the answer for v_d in the case under consideration:

$$v_d=\frac{vv_u}{(v+2v_u)}. \qquad (13)$$

One sees immediately that $v_d<v,v_u$. Inserting this expression into Eq. (12) we find that the intervals of the variable z for $u(z)$ in Eqs. (11) and (12) are the same.

Inserting Eq. (13) into Eq. (9) we obtain

$$w\left(-3\frac{(v+v_u)^2}{vv_u}\frac{r_0}{\theta_f}<z<-\frac{(v+v_u)^2}{vv_u}\frac{r_0}{\theta_f}\right)$$

$$=\frac{v+v_u}{2v_u}r_0+\frac{(v+2v_u)}{2(v+v_u)}\theta_fz. \qquad (14)$$

At last, substitution of Eq. (2) into Eq. (8a) gives, accounting for already known answers for $w(z)$ Eqs. (10) and (14) in different intervals of z, the following expressions for $u(z)$:

$$u\left(\frac{(v+v_u)}{v_u}\frac{r_0}{\theta_f}<z<\frac{(v+v_u)^2}{v_u^2}\frac{r_0}{\theta_f}\right)=r_0+\frac{v_u(v+2v_u)}{(v+v_u)^2}\theta_fz \qquad (15)$$

and

$$u\left(\frac{(v+v_u)^2}{v_u^2}\frac{r_0}{\theta_f}<z<3\frac{(v+v_u)^2}{v_u^2}\frac{r_0}{\theta_f}\right)$$

$$=\frac{3}{2}r_0+\frac{v_u(2v+3v_u)}{2(v+v_u)^2}\theta_fz. \qquad (16)$$

Combining Eqs. (11) and (15) we get

$$u\left(0<z<\frac{(v+v_u)^2}{v_u^2}\frac{r_0}{\theta_f}\right)=r_0+\frac{v_u(v+2v_u)}{(v+v_u)^2}\theta_fz. \qquad (17)$$

Eqs. (16), (17), (10), and (14) present a full answer for $u(z)$ and $w(z)$. One may see easily that they are bounded continuously, so our choice of t_3 was correct. Inserting Eqs. (16), (17), (10), and (14) into Eq. (2) we obtain a full solution of the problem for $t_1<t<t_3$:

$$r(v_u(t-t_2)<x<0,t>t_1)=r_0+\frac{v_u}{v+v_u}\theta_fx, \qquad (18a)$$

$$r\left(-\frac{(v+v_u)^2}{v_u(v+2v_u)}\frac{r_0}{\theta_f}-\frac{vv_u}{v+2v_u}t<x<-v(t-t_2),v_u(t-t_2)\right)$$

$$=\frac{r_0}{2}+\frac{v_u(v+2v_u)}{2(v+v_u)^2}\theta_f\left(x+\frac{vv_u}{v+2v_u}t\right), \qquad (18b)$$

$$r(-v(t-t_2)<x<0,t<t_3)=r_0+\frac{v_u}{2(v+v_u)}\theta_fx, \qquad (18c)$$

$$h(v_u(t-t_2),-vt<x<0)=\frac{v_u}{(v+v_u)}\theta_f(x+vt), \qquad (18d)$$

$$h\left(-\frac{(v+v_u)^2}{v_u(v+2v_u)}\frac{r_0}{\theta_f}-\frac{vv_u}{v+2v_u}t<x\right.$$

$$\left.<-v_u(t-t_2),0,t<t_3\right)$$

$$=\frac{v+v_u}{2v_u}r_0+\frac{(v+2v_u)}{2(v+v_u)}\theta_f\left(x+\frac{vv_u}{v+2v_u}t\right). \qquad (18e)$$

These solutions are shown in Figs. 1(b–e) [see also Fig. 2, the region $t_1<t<t_3$ in which trajectories of the front — a lower segmented line — and the breaks of $r(x,t)$ and $h(x,t)$ are depicted]. The solutions (18a) and (18d) are defined in region 1 of Fig. 2, the solution (18b) is defined in region 2 of

the figure, the solution (18c) is defined in region 3, and the solution (18e) for $h(x,t)$ is defined in both regions 2 and 3.

The evolution looks as the following. At the instant t_1 new linear parts of r and h appear at the front point $x = -[(v+v_u)/v_u]r_0/\theta_f$. The new front moves to the left with the velocity v_d, breaks of $r(x,t)$ and $h(x,t)$ move to the right with the velocity v_u—a solid line $x=v_u(t-t_2)$ in Fig. 2. The slope of this part of the static grain distribution $h(x,t)$ is higher than it was at the first stage but lower than the critical slope. The breaks approach the $x=0$ point at $t=t_2$. Then a new time-independent linear part of $r(x)$ appears close to the wall. The left part proceeds to move to the left, so we see a break moving downhill — a dashed line $x=-v(t-t_2)$ in Fig. 2. Its velocity equals v. Note that, unlike $r(x,t)$, $h(x,t)$ has no breaks for $t_2<t<t_3$. At the instant t_3 the break of $r(x,t)$ overtakes the front. That is possible, since the velocity of the front is lower than the downhill velocity of the break, $v_d<v$.

Therefore we confirm the general picture of the evolution predicted in Sec. I: there exist breaks of both $r(x,t)$ and $h(x,t)$ moving uphill with the velocity v_u, and only the break of $h(x,t)$ is moving downhill with the velocity v. When the breaks approach a wall or the break of $h(x,t)$ overtakes the front, new linear parts of the solutions appear.

From our solutions Eq. (18), we obtain new initial conditions for $r(x,t)$ and $h(x,t)$ at the moment $t=t_3$. Then we may repeat the described procedure for the next time interval $t_3<t<t_5$, etc. However, now, when we understand the structure of the solution, one may again simplify the calculations. Instead of calculating the functions $u(z)$ and $w(z)$ one may proceed directly with $r(x,t)$ and $h(x,t)$. Areas of different linear parts of solutions are triangular regions in Fig. 2 separated one from each other by solid or dashed lines — trajectories of the breaks. Let us suppose that the solution is known in the region $(0,0)$ — (x_1,t_1) — $(0,t_2)$, i.e., in region 1 in Fig. 2. One may sew easily an unknown linear solution in the adjusted triangular region 2 together with the known one along the line $x=-v(t-t_2)$. It is easy to find coefficients of linear terms and, therefore, the front velocity. From this values one finds the shape (x_1,t_1) — $(0,t_2)$ — (x_2,t_3) of region 2 in which the new part of the solution is defined (see Fig. 2). Then we repeat the described procedure for the next triangle, etc.

After these simple but rather tedious calculations we obtain the total solution consisting of linear parts, a structure of which one can see from Figs. 1 and 2 and following equations. The lowest segmented line in Fig. 2 shows the dependence on time of the front position. Coordinates of the segments are:

$$x(t_{2m-1}<t<t_{2m+1})=-\frac{m(m+1)}{2}\frac{(v+v_u)^2}{v_u\mathcal{V}_m}\frac{r_0}{\theta_f}-\frac{vv_u}{\mathcal{V}_m}t,$$

$$m=0,1,2,\dots \qquad (19)$$

(for $m=0$ the time interval is $0<t<t_1$) with $\mathcal{V}_m\equiv mv+(m+1)v_u$. The particular times shown in Fig. 2 are

$$t_{2m-1}=\frac{m}{2}\frac{(v+v_u)[(m-1)v+(m+1)v_u]}{vv_u^2}\frac{r_0}{\theta_f},$$

$$t_{2m}=\frac{m(m+1)}{2}\frac{(v+v_u)^2}{vv_u^2}\frac{r_0}{\theta_f},$$

$$m=1,2,\dots \qquad (20)$$

The front coordinates corresponding to times t_{2m-1} at which the front velocity changes its value are

$$x(t_{2m-1})=-m\frac{(v+v_u)}{v_u}\frac{r_0}{\theta_f}, \qquad m=1,2,\dots \qquad (21)$$

Hence, these points are arranged periodically. Two other types of lines are shown in Fig. 2: solid lines $x=v_u(t-t_{2m})$ and dashed lines $x=-v(t-t_{2m})$, $m=1,2,\dots$. The lines of the first type depict the uphill movement of the breaks of both profiles $r(x)$ and $h(x)$ [Fig. 1(c)]. The lines of the second type show the downhill movement of the break of the profile $r(x)$ with the velocity v [see Fig. 1(e)].

Solutions for all regions of Fig. 2, connected at these lines, look as

$$r(-v(t-t_{2m-2}),v_u(t-t_{2m})<x<0)$$

$$=r_0+\frac{v_u}{m(v+v_u)}\theta_f x,$$

$$r\left(-\frac{m(m+1)}{2}\frac{(v+v_u)^2}{v_u\mathcal{V}_m}\frac{r_0}{\theta_f}-\frac{vv_u}{\mathcal{V}_m}t\right.$$

$$\left.<x<-v(t-t_{2m}),v_u(t-t_{2m})\right)$$

$$=\frac{r_0}{2}+\frac{1}{m(m+1)}\frac{v_u\mathcal{V}_m}{(v+v_u)^2}\theta_f\left(x+\frac{vv_u}{\mathcal{V}_m}t\right),$$

$$m=1,2,\dots \qquad (22)$$

$$h(v_u(t-t_{2m+2})<x<v_u(t-t_{2m}),0)$$

$$=\frac{m}{2}\frac{v+v_u}{v_u}r_0+\frac{\mathcal{V}_m}{(m+1)(v+v_u)}\theta_f\left(x+\frac{vv_u}{\mathcal{V}_m}t\right),$$

$$m=0,1,2,\dots \qquad (23)$$

Here, we set $t_0=0$. Inequalities in the right hand parts of Eqs. (22) and (23) define the areas of validity of the solutions (see Fig. 2). Equations (22) and (23) describe totally the sandpile evolution, see Fig. 1(b-e).

From Eq. (23) and Fig. 2, a space periodicity of the process is evident: general shapes of the profiles $r(x)$ and $h(x)$ are repeated each time the front moves by $(v+v_u)r_0/(v_u\theta_f)$ to the left. In fact, the pile is increased layer by layer, and the expression for the width of these layers is the same as the one for the width of stratified layers of different fractions in a two-component sandpile (that was obtained in papers [11,14]). One sees from Eq. (23) that heights of the pile at times t_{2m} are also periodic in m:

$$h(x=0,t_{2m})=m\frac{v+v_u}{v_u}r_0, m=0,1,2,\dots \qquad (24)$$

It follows from Eq. (23) that the slope, $\theta_h \equiv \partial h / \partial x$, of the static part of the pile will approach its critical value only at infinite time:

$$\theta_h(t_{2m-3} < t < t_{2m}) = \left(1 - \frac{v}{m(v+v_u)}\right)\theta_f, \quad m=1,2,\ldots$$
(25)

(for $m=1$, $0 < t < t_2$) (see Fig. 3). At $t_{2m-1} < t < t_{2m}$, $m = 1,2,\ldots$ there are two different slopes for two parts of the profile $h(x)$. For $t_{2m} < t < t_{2m+1}$, $m = 0,1,2,\ldots$ all the profile has the same slope. Thus, for long times $t \gg [(v + v_u)^2/(v v_u^2)] r_0/\theta_f$ the slope behaves as

$$\theta_h \cong \left(1 - \frac{v}{v_u}\sqrt{\frac{r_0}{2 v \theta_f t}}\right)\theta_f,$$
(26)

and θ_h relaxes slowly to its final value θ_f by a power law.

A slope of the whole pile — including both static and rolling parts, $\theta_{h+r} \equiv \partial(h+r)/\partial x$, depends on time in the following way:

$$\theta_{h+r}(t_{2m-2} < t < t_{2m}) = \left(1 + \frac{v_u - v}{m(v+v_u)}\right)\theta_f,$$

$$\theta_{h+r}(t_{2m-1} < t < t_{2m+1})$$

$$= \left(1 + \frac{v_u - v}{m(v+v_u)} + \frac{v^2}{m(m+1)(v+v_u)^2}\right)\theta_f, \quad (27)$$

$$m = 1,2,\ldots$$

(see Fig. 3) and approaches the final value θ_f from above at long times. It behaves asymptotically as

$$\theta_{h+r} \cong \left(1 + \frac{v_u - v}{v_u}\sqrt{\frac{r_0}{2 v \theta_f t}}\right)\theta_f$$
(28)

[compare with Eq. (26)].

At long times, the coordinate of the front is $x \cong \sqrt{2 v r_0/\theta_f}$, and its velocity tends to $-\sqrt{v r_0/(2\theta_f t)}$. Two last relations are obvious. Indeed, at long times the slope of a pile is close to its critical value. The relative part of rolling grains is small. After substitution of a linear function with the critical slope into the condition of conservation of grains Eq. (3) we obtain last relations immediately.

B. Time dependent input flow

As we have noted in Sec. I, space periodicity of the pile evolution may disappear if an input flow increases with time. Let us consider the simplest example of a linear time dependence: $r_0(t) = v_0 t$ (the constant v_0 has dimensionality of a velocity). Now there are not any jumps of profiles at the front point, since the growth of r_0 starts from zero and an initial state is empty, so the problem is simpler than that considered above.

Instead of trying to solve the problem directly, one may proceed in the following way. First we suppose that the velocity of the front is constant. Let us call it v_d again. From a moving boundary condition $h(x=-v_d t, t \geqslant 0) = 0$ at the front point accounting for Eq. (2) we obtain $w(z)$. Then from

another moving boundary condition $r(x=-v_d t, t \geqslant 0) = 0$ and Eq. (2) one may get $u(z)$. Inserting both answers into the condition $r(x=0,t) = v_0 t$ we (i) verify that the choice of a constant front velocity is right and (ii) find v_d as a function of v_0.

We write out the answers immediately and check them by substituting in Eq. (1). The profiles of moving and static grains look as

$$r = \frac{v_0}{v_d}(x + v_d t),$$

$$h = \frac{v_u}{v_d + v_u}\theta_f(x + v_d t)$$
(29)

(see Fig. 4), where the front velocity is

$$v_d = \frac{1}{2}\frac{v_u - v}{1 + \theta_f v_u/v_0}\left[\sqrt{1 + 4\frac{v v_u}{(v_u-v)^2}\left(1 + \theta_f\frac{v_u}{v_0}\right)} - 1\right].$$
(30)

In the figure, the evolution of the pile in the situation under consideration is depicted. One may see that the relation between amounts of rolling and static grains depends considerably on the rate of the input flow increase. As it follows from Eqs. (29) and (30), when $v_0 \gg v_u \theta_f$,

$$v_d \cong v,$$

$$r \cong \frac{v_0}{v}(x + vt),$$

$$h \cong \frac{v_u}{v+v_u}\theta_f(x+vt),$$
(31)

and the amount of rolling grains is much higher than the amount of static grains [Fig. 4(a)]. The front velocity approaches its highest possible value, and rolling grains have no time to converse into static ones. For a low c

$$v_d \cong \sqrt{\frac{v_0 v}{\theta_f}},$$

$$r \cong \sqrt{\frac{v_0 \theta_f}{v}}\left(x + \sqrt{\frac{v_0 v}{\theta_f}}\,t\right),$$
(32)

$$h \cong \theta_f\left(x + \sqrt{\frac{v_0 v}{\theta_f}}\,t\right),$$

so the front velocity tends to zero, the slope of the distribution of static grains is nearly critical, and the relative amount of rolling grains is small [Fig. 4(b)].

The amounts of rolling and static grains in a pile are equal for the following particular value of the rate of the input flow growth:

$$v_0^* = \frac{v v_u}{v + 2 v_u} \theta_f \qquad (33)$$

[compare with Eq. (13) from Sec. II A]. In this particular case, the front velocity is $v_d^* = v/2$.

III. SANDPILE EVOLUTION STARTING FROM CRITICAL STATE

Now, let us switch on the input flow of grains when there is already a pile with the critical slope. Thus, the initial condition for h is $h(x, t = 0) = \theta_f(x + d)$, where d is the initial horizontal size of a pile. All other initial and boundary conditions are the same as in Sec. II A (the input flow is time-independent).

Now one can proceed with the calculations similar to those that were made in Sec. II to obtain the solutions describing the evolution of the pile. We omit the tedious calculations and present immediately results for the case under consideration. Main answers are presented schematically in Fig. 5 in which the distributions of rolling grains $r(x)$ and the static ones $h(x)$ are shown in several successive moments. (Note that scales change from one figure to another.) Trajectories of the front and the breaks on the t,x-plane are shown in Fig. 6.

Let us comment the contents of Fig. 5, since it looks rather intricate. At the initial moment rolling grains are absent, and the angle of the profile of static grains is critical. After we switch on an external flow, a step of rolling grains starts to descend downhill the critical profile of static grains [Fig. 5(a)]. Rolling grains do not convert to the static ones.

After the step approaches the last point of the profile of static grains [Fig. 5(b)] it proceeds to move to the left with the same velocity v, but the height of the jump starts to decrease with time [Fig. 5(c)]. A new linear part of $r(x)$ emerges and begins to move uphill with the velocity v_u. (Recall that all linear parts of profiles move as a whole or stay without movement. They can not change their slope with time.) A new more gently sloping part of $h(x)$ simultaneously appears and begin to move to the left with the velocity v, so a break of $h(x)$, coinciding with the break of $r(x)$, moves uphill with the velocity v_u.

After the decreasing jump at the front of $r(x)$ disappears [Fig. 5(d)] the linear part of $r(x, t)$ proceeds to move uphill with the velocity v_u [Fig. 5(e)]. The corresponding part of $h(x)$ also proceeds to move uphill. Behind it, a new static part of the profile with the critical slope emerges, so, in fact, an inclined step moves uphill.

When its first point touches a wall [Fig. 5(f)], a new static linear part of $r(x)$ appears to satisfy the boundary condition $r(x = 0, t) = r_0$ [Fig. 5(g)]. An old part of $r(x)$ proceeds to move uphill, so a new break of $r(x)$, that moves downhill with the velocity v, emerges. Evolution of $h(x, t)$ proceeds as before. At some instant [Fig. 5(h)] the last point of $r(x)$ and the break of $r(x)$ meet. New linear part of $r(x)$ emerges and starts to move downhill [Fig. 5(i)]. The $h(x)$ profile evolves in the same way as before.

When all the $h(x)$ profile appears to be critical [Fig. 5(j)], the whole $r(x)$ distribution is linear. This inclined profile starts to move down the critical one [Fig. 5(k)]. The only difference with the instant depicted in Fig. 6(a) is that the

step of rolling grains is inclined now. After its first point approaches the last left point of the critical pile [Fig. 5(l)] new linear parts of $r(x)$ and $h(x)$ emerge that move to the left with some velocity $v_d < v, v_u$ [Fig. 5(m)]. Two breaks of $r(x)$ emerge that move one to each other.

After the breaks meet [Fig. 5(n)], two new breaks of $r(x)$ emerge that move away one from each other [Fig. 5(o)] while $h(x, t)$ proceeds the previous evolution. The moving downhill break of $r(x)$ overtakes the front at some moment [Fig. 5(p)]; after that an inclined step starts to climb uphill leaving after itself a new static critical slope part [Fig. 5(q)]. Afterwards the evolution proceeds in the same way.

Thus, one sees that the evolution of a pile in this case looks more complicated than in Sec. II. Nevertheless, as it is evident from Fig. 6, in which all trajectories of the front and breaks of $r(x, t)$ and $h(x, t)$ are shown, a general similarity remains.

Let us describe Fig. 6. The solid lines $x = v_u(t - t_{2m}^-), v_u(t - t_{2m}), \ m = 0, 1, \ldots$ show trajectories of the $r(x, t)$ and $h(x, t)$ breaks moving uphill (we introduce these notations for the instants of time to keep a tie with the corresponding notations in Sec. II). The dashed lines $x = -v(t - t_{2m}^-), \ -v(t - t_{2m}), \ m = 0, 1, \ldots$ and $x = -vt$ depict trajectories of the $r(x, t)$ break moving downhill. Comparing with Fig. 2 from the previous section one sees that the corresponding lines for the break trajectories are splitted now, and static segments of the front trajectory appear. The expressions for particular times shown in Fig. 6 are

$$t_{2m-2} = m \frac{v + v_u}{v v_u} d + \frac{(m-1)m}{2} \frac{(v + v_u)^2}{v v_u^2},$$

$$t_{2m}^- = m \frac{v + v_u}{v v_u} d + \frac{m(m+1)}{2} \frac{(v + v_u)^2}{v v_u^2}, \qquad (34)$$

$$m = 1, 2, \ldots$$

— for the instants when the breaks of $r(x, t)$ and $h(x, t)$ reach a wall — and

$$t_{2m-1} = \frac{\mathcal{V}_m}{v v_u} d + \frac{m}{2} \frac{v + v_u}{v v_u^2} [(m-1)v + (m+1)v_u] \frac{r_0}{\theta_f},$$

$$t_{2m+1}^- = \frac{\mathcal{V}_m}{v v_u} d + \frac{m+1}{2} \frac{v + v_u}{v v_u^2} [mv + (m+2)v_u] \frac{r_0}{\theta_f}, \qquad (35)$$

$$m = 0, 1, \ldots$$

— for the times at which the break of $r(x, t)$ overtakes the front.

The trajectory of the front can be described by the following relation

$$x(t_{2m+1}^- < t < t_{2m+1}) = -d - (m+1)\frac{v+v_u}{v_u}\frac{r_0}{\theta_f},$$

$$x(t_{2m-1} < t < t_{2m+1}^-) = -\frac{m(m+1)}{2}\frac{(v+v_u)^2}{v_u \mathcal{V}_m}\frac{r_0}{\theta_f} - \frac{vv_u}{\mathcal{V}_m}t, \tag{36}$$

$$m = 0,1,\ldots$$

and $x(0 < t < t_{-1}) = -d$. Note that the second relation of Eq. (36) coincides with Eq. (19). Thus, at the corresponding instants, the front velocities are the same in both cases. The first relation of Eq. (36) again demonstrates the space periodicity of the process.

Time intervals for which the front is motionless do not change with time:

$$t_{2m-1} - t_{2m-1}^- = \frac{v+v_u}{vv_u}d, \qquad m = 1,2,\ldots \tag{37}$$

These intervals appear because of a new initial condition, so even at long times one may observe the influence of the initial condition. Nevertheless, the duration of the other time intervals, when the front moves, increases with time. Thus, the relative effect of the initial conditions decreases with time.

We shall need only the following result for h:

$$h[v_u(t - t_{2m+2}^-) < x < v_u(t - t_{2m+2})]$$

$$= \theta_f\left(x + d + (m+1)\frac{v+v_u}{v_u}\frac{r_0}{\theta_f}\right),$$

$$h[v_u(t - t_{2m+2}^-) < x < v_u(t - t_{2m})]$$

$$= \frac{m}{2}\frac{(v+v_u)}{v_u}r_0 + \frac{\mathcal{V}_m}{(m+1)(v+v_u)}\theta_f\left(x + \frac{vv_u}{\mathcal{V}_m}t\right), \tag{38}$$

$$m = 0,1,\ldots.$$

One sees that these relations are valid for different strips $v_u(t - t_{2m+2}^-) < x < v_u(t - t_{2m+2})$ and $v_u(t - t_{2m+2}^-) < x < v_u(t - t_{2m})$ of the x,t-plane. The first relation of Eq. (38) describes parts of the distribution of static grains with the critical slope, and the second relation of Eq. (38) coincides with the corresponding answer Eq. (23) for the first case.

The slope angles of the profiles of static grains obtained from Eq. (38) are shown in Fig. 7, which demonstrates most naturally the relaxation to the critical state. Unlike the previous section (see Fig. 2), in this case, segments with the critical angle are present. As can be seen from Fig. 7, different segments may overlap, and the profile may have two or even three parts with different slopes simultaneously (see also Fig. 6). Long time asymptotes of the pile slope are the same as in Sec. II.

IV. CONCLUSIONS

In summary, in the case of the thick flow regime, we have shown that space periodicity takes place during a sandpile evolution even for the one-component pile. The pile spills during a repeating process if an income flow is constant: grains are piling layer by layer. The thickness of the layers coincides surprisingly with the thickness of stratified layers at the two-component sandpile problem [11,14]. Thus, in the one-component pile, we found a clear precursor of the stratification phenomena.

We have found very reach behavior using the most simple and clear approach [24] admitting an easy analytical treatment. The slope of the pile goes to its final critical value after complex relaxation, which is a long discontinuous process. Indeed, in the case of a constant income flow, the pile slope may change only discontinuously (see Figs. 2 and 6). At long times the deviation of the slope from its critical value is proportional to $\sqrt{r_0/t}$.

The pile evolution from different initial conditions looks very similar at long times but even at such times one can observe some difference (compare Figs. 2 and 6 for two most natural situations: the sandpile evolution starting from an empty state and the evolution starting from a critical state).

The space periodicity disappears if an input flow grows linearly with time. In such a situation the sandpile front moves with a constant velocity all the time, and the pile evolves continuously without any peculiarities (see Fig. 4).

Formally speaking, we studied only the one-dimensional problem, but an ordinary three-dimensional sandpile with an axial symmetry (that means that sand is pouring on an infinite horizontal plane at a single point) may be described by the same equations as Eqs. (1) (the coordinate x means the distance from the center). Thus, our results also stay valid for such a pile if one applies the phenomenological description [18,24].

Aradian, Raphaël, and de Gennes [29] have recently introduced a dependence of the downhill velocity v on r in the frames of the thick flow regime ($v_u =$ const). The profile shapes in that case become nonlinear but a general picture of a sandpile evolution will hardly change, since the structure of the evolution equations is the same — in fact, the second equation of Eqs. (1) is independent of the first one.

The following questions remain open. How crucially do our results depend on the used, maybe oversimplified, phenomenological approach? Are they really of general significance? Does the sandpile evolution in a thin flow regime differ essentially from the described one? It is apparently impossible to answer these questions within the present approach, however the last results [32] and [33] on granular flows in thin and intermediate regimes let one hope to solve these beautiful problems.

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